Adiabatic Invariants

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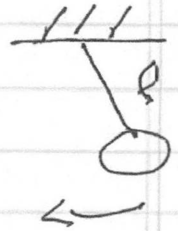
→ 

$\ell = \ell(t)$

s/t $\boxed{\frac{\dot{\ell}}{\ell} < \sqrt{g/\ell}}$ → 2 time scale

→ 'adiabatic' variation of parameter.

→ How describe?

→ Now $\ddot{\theta} + \frac{g}{\ell(\cancel{t})}\theta = 0$

essence is oscillator with slowly varying parameter

$$\ell(t) = \ell(\underbrace{\epsilon t}_{\text{slow}})$$

$$\ddot{\theta} + \frac{g}{\ell(\epsilon t)}\theta = 0$$

$\epsilon t = \tau$

$\Rightarrow dt = \frac{1}{\epsilon}\, d\tau$

$d/dt =$ [redacted] $\epsilon \frac{d}{d\tau}$

$$\ddot{\theta} + \frac{g}{\epsilon^2 \ell(\tau)}\theta = 0 \Rightarrow \boxed{\ddot{\theta} + \frac{\omega(\tau)^2}{\epsilon^2}\theta = 0}$$

generically points toward WKB.

i.e. generic:

$$\ddot{x} + \omega^2(\epsilon t)\, x = 0$$

$\Rightarrow \tau = \epsilon t$

$$\frac{d^2 x}{d\tau^2} + \frac{\omega^2(\tau)}{\epsilon^2}\, x = 0$$

→ A different look at adiabatic theory . . .

One might forego canonical formalism, and simply investigate an oscillator with slowly varying frequency

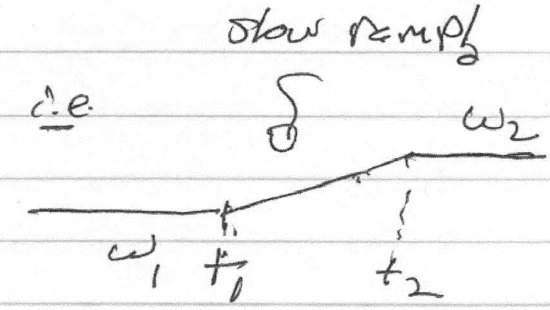
i.e.

$$\ddot{x} + \omega^2 x = 0 \quad \Rightarrow$$

$$\ddot{x} + \omega^2(\epsilon t)\, x = 0$$

slowly varying frequency

i.e.



i.e.

$$\frac{1}{\omega}\frac{d\omega}{dt} \sim O(\epsilon\omega) \ll 1$$

⇒ expect, on basis of previous discussion,

I ⇒ adiabatic invariant

i.e. if $a \equiv$ oscillator amplitude, then

$$I = E/\omega = \tfrac{1}{2} m\omega^2 a^2/\omega \simeq m\omega a^2$$

as const

→ action!

Need show action is const, adiabatic invariant.

Now, for slowly varying ω, can solve by WKB!

now, $\boxed{\epsilon t = \tau}$

$$\frac{d^2 X}{d\tau^2} + \frac{\omega^2(\tau)}{\epsilon^2} X = 0$$

$$X(\tau) = a_0 e^{i\phi(\tau)/\epsilon}$$

where: $\phi = \phi_0 + \epsilon \phi_1 + \ldots$

eikonal correction $\to$ ampl.

$$\frac{d}{d\tau}\left(a_0 \frac{i\dot{\phi}(\tau)}{\epsilon} e^{i\phi(\tau)} \right) + \frac{\omega(\tau)^2}{\epsilon^2} a_0 e^{i\phi(\tau)} = 0$$

$$\left(-\frac{\dot{\phi}^2}{\epsilon^2} + \frac{i\ddot{\phi}(\tau)}{\epsilon} \right) a_0 e^{i\phi} + \frac{\omega(\tau)^2}{\epsilon^2} a_0 e^{i\phi(\tau)} = 0$$

need to $O(1/\epsilon)$

$\Rightarrow$

$$\left(-\frac{(\dot{\phi}_0 + \epsilon \dot{\phi}_1)^2}{\epsilon^2} + \frac{i\ddot{\phi}_0(\tau)}{\epsilon} \right) + \frac{\omega(\tau)^2}{\epsilon^2} = 0$$

$$-\frac{\dot{\phi}_0^2}{\epsilon^2} + \frac{\omega(\tau)^2}{\epsilon^2} = 0$$

$$\dot{\phi}_0(\tilde{t}) = \omega(\tilde{t})$$

$$\phi_0(\tilde{t}) = \int \omega(\tilde{t})\, d\tilde{t}$$

For next order correction.

$$-\frac{2\dot{\phi}_0\dot{\phi}_1}{\epsilon} + \frac{i\ddot{\phi}_0}{\epsilon} = 0$$

$$\dot{\phi}_1 = i\ddot{\phi}_0 / 2\dot{\phi}_0$$

$$= \frac{i}{2}\frac{d}{dt}\ln\left(\dot{\phi}_0(\tilde{t})\right)$$

$$\phi_1 = \frac{i}{2}\ln\left(\dot{\phi}_0(\tilde{t})\right)$$

$$= \frac{i}{2}\ln\left(\omega(\tilde{t})\right)$$

So

$$x(t) = a_0 e^{i\phi(\tilde{t})/\epsilon}$$

$$= a_0 e^{i\int \frac{\omega(\tilde{t})}{\epsilon} dt} e^{i\frac{i}{2}\ln(\omega(\tilde{t}))}$$

$$= \underline{a_0} e^{i\int \frac{\omega(\tilde{t})}{\epsilon} dt} e^{-\frac{\ln}{2}\omega(\tilde{t})}$$

14.

$\Rightarrow \quad x(t) = \underline{a_0}\, e^{i\int \frac{\omega}{\epsilon} dt}\, e^{-\frac{1}{2}\ln\omega}$

$= \frac{a_0}{\sqrt{\omega}}\, e^{i\int \frac{\omega}{\epsilon}}$

$d\tau = \epsilon\, dt$

re-scaling, $\quad t = \tau/\epsilon \; ; \quad \omega(\epsilon t)$

$$\boxed{x(t) = \frac{a_0}{\sqrt{\omega}}\, e^{i\int \omega(\epsilon t)\, dt}}$$

WKB soln.

and can observe:

$\overline{\omega x^2} = \omega\, \overline{x^2} = \omega \frac{a_0^2}{\omega} = \text{const}!$

(cycle) action $\sim \overline{\omega^2 x^2}/\omega$

$\Rightarrow$ Action is invariant, due to frequency modulation of amplitude!

check: $\quad I = \frac{1}{2\pi}\oint p\, dq$

$= \frac{1}{2\pi}\oint p\, dx$

$= \frac{1}{2\pi}\oint m\dot{x}\, dx = \frac{1}{2\pi}\oint m\dot{x}\,\dot{x}\, dt$

$$I = \frac{1}{2\pi} \oint_{\omega^{-1}} m\dot{x}^2 \, dt$$

$$x(t) = \frac{a_0}{\sqrt{\omega}} \cos(\omega t + \delta)$$

$$\dot{x} = -a_0 \sqrt{\omega} \sin(\omega t + \delta)$$

$\theta = \omega t$ $\qquad \delta \to 0 \qquad \omega^2 / (\sqrt{\omega})^2$

$d\theta = \omega dt$

$$I = \frac{1}{2\pi} \oint p \, dq = \frac{1}{2\pi} \oint d\theta \, a_0^2 \, \omega \sin^2\theta \, \frac{d\theta}{\omega}$$

$$= \frac{1}{2} a_0^2 \to \text{real const.!}$$

⇒ the message:

* — adiabatic invariants basically a consequence of WKB approximation (time scale sep!)
* — WKB would lead one to adiabatic invariance of action, even if did not realize it.
* — need retain WKB correction beyond pure eikonal for freq. modulation of amplitude ⇒ essential.

→ Adiabatic Invariants [and Action - Angle Variables]

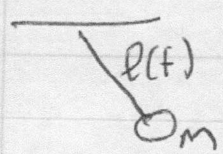
i) Adiabatic Invariants — Formal Approach (Bounded phase space ⟺ QPO)

→ Consider finite motion in 1D. Motion characterized by λ parameter, such that:

$$\frac{1}{\lambda}\frac{d\lambda}{dt} \ll \frac{1}{\tau}$$

↳ period of motion

i.e.



$$\frac{1}{\ell}\frac{d\ell}{dt} < \sqrt{g/\ell}$$

pull on string

thus, $\dot{E}$ will be "small / slow" (i.e. $H = H(\lambda(t), p, q)$)

parametric dependence,

Now, $\dfrac{dE}{dt} = \dfrac{\partial H}{\partial t} = \dfrac{\partial H}{\partial \lambda}\dfrac{d\lambda}{dt}$

as λ varies slowly compared to $\omega_0 = 1/\tau$, can average over t on fast scales, i.e.

$$\frac{d\overline{E}}{dt} = \overline{\frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt}} \cong \overline{\frac{\partial H}{\partial \lambda}}\,\frac{d\lambda}{dt}$$

break avg. on basis time scale separation

↳ avg. over motion q, p → fast

$- \omega$

$- \dot{\omega}/\omega$

where $\overline{A} = \frac{1}{T}\int^{\tau} dt\, A(t)$ → holding E, λ fixed!

average on fast timescale

$$\Rightarrow \quad \overline{\frac{\partial H}{\partial \lambda}} = \frac{1}{T}\int^{\tau} \frac{\partial H}{\partial \lambda}\, dt$$

Now,

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$dt = \int dq / \partial H/\partial p$$

∴ can take $\int dt \rightarrow \oint \frac{dq}{\partial H/\partial p}$

$\oint$ ↔ complete circuit orbit

so finally,

$$\frac{d\overline{E}}{dt} = \frac{d\lambda}{dt}\left\{\frac{\oint(\partial H/\partial\lambda)\, dq \,/\, (\partial H/\partial p)}{\oint dq/(\partial H/\partial p)}\right\}$$

evt

$$\equiv \frac{d\lambda}{dt}\left\langle \frac{\partial H}{\partial \lambda}\right\rangle_{\tau}.$$

Now: – integrations must be performed for fixed, given value of λ (i.e. $\dot{\lambda}/\lambda \ll \omega_0$)
– on such path (n.b. why "path" of interest !), $H = E$ and $p = p(q; E, \lambda)$

$$\therefore \quad H(p, q, \lambda) = E$$

{path for E const.

$$\frac{\partial H}{\partial \lambda} + \frac{\partial H}{\partial p}\frac{\partial p}{d\lambda} = 0$$

$$\Rightarrow \quad \frac{\partial H/\partial \lambda}{\partial H/\partial p} = -\frac{\partial p}{\partial \lambda}$$

plug in previous

$$\therefore \quad \frac{d\bar{E}}{dt} = \frac{d\lambda}{dt}\frac{\oint -(\partial p/\partial \lambda)\, dq}{\oint dq\, \partial p/\partial E}$$

$(1/\partial H/\partial p = \partial p/\partial E)$ (fixed λ)

so, re-writing:

$$\frac{d\bar{E}}{dt}\oint dq\, \partial p/\partial E + \frac{d\lambda}{dt}\oint (\partial p/\partial \lambda)\, dq = 0$$

$$\Rightarrow \oint_{\substack{E,\lambda \\ \text{fixed}}} dq \left\{ \frac{\partial p}{\partial E}\frac{d\bar{E}}{dt} + \frac{\partial p}{\partial \lambda}\frac{d\lambda}{dt} \right\} = 0$$

$$\Rightarrow \boxed{\overline{dI/dt} = 0}$$

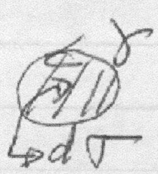
where $I = \oint_{\substack{E,\lambda \\ \text{fixed}}} \frac{p\,dq}{2\pi}$ → { integral taken over path for fixed, given E, λ ! }

∴ → $\boxed{\begin{array}{l} I \text{ const. as } \lambda \text{ varies !} \\ \therefore\ I \text{ adiabatic invariant} \end{array}}$ (abbreviated action along approx. traj.)

→ in general (including higher dimensions)

$$I_c = \oint_{\gamma} \underline{p}\cdot d\underline{q} = \iint_{\sigma} d\underline{p} \wedge d\underline{q}$$ { Liouville Thm, again }

is Poincares relative integral invariant (γ closed curve, enclosing σ)



I_c is exact invariant.

$$I = \oint p\, dq$$

so $\left. I = I_c \right|_{E,\,\lambda \text{ constant}}$

is approximation to Poincare invariant

for $\dot{\lambda}/\lambda < \omega_0$. { Hence adiabatic invariant.

long time scales

Now, adiabatic invariant:

$$\underset{\lambda, E \text{ fixed}}{I = \oint dq\, p/2\pi} \quad \rightarrow \text{ what is it?}$$

so $I = I(E)$

$$= \oint \frac{p\, dq}{2\pi}$$

$$2\pi \frac{\partial I}{\partial E} = \oint \frac{\partial p}{\partial E} dq = \oint \frac{dq}{\partial H/\partial p} = \tau$$

$\therefore$ $\boxed{\partial I/\partial E = 1/\omega}$

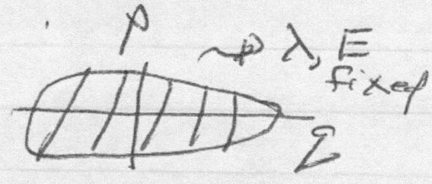
$\boxed{\partial E/\partial I = \omega}$

Now, of course: adiabatic invariant has geometrical significance.

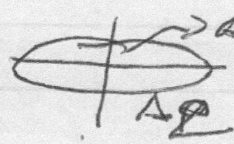
$$I = \oint_{E,\lambda} \frac{p\,dq}{2\pi} = \iint_{E,\lambda} \frac{dp\,dq}{2\pi}$$

$\therefore$ I corresponds to enclosed area!

i.e.



e.g. $$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 = E$$



$$\Delta p = (2mE)^{1/2}$$
$$\Delta q = (2E/m\omega^2)^{1/2}$$

$$\text{Area} = \pi\,\Delta q\,\Delta p = 2\pi E/\omega$$

$\boxed{I = E/\omega}$ $\Rightarrow$ $\left\{\right.$ for oscillator adiabatic invariant is <u>action</u>, E/ω.

$\therefore$ $\dot{\ell}/\ell < \omega_0$ $\Rightarrow$

$E \sim \omega \sim \sqrt{g/\ell}$

Physics 200 A, B Mechanics

→ Adiabatic Invariants: Review

→ if $H = H(p, q, \lambda(t))$

$\lambda(t)$ → parametric dependence

with a) periodic motion, for fixed λ.

b.) $\frac{1}{\lambda}\frac{d\lambda}{dt} < \omega$

$\frac{1}{\lambda}\frac{d\lambda}{dt}$ → rate of change of parameter

ω → motion frequency

then $I_\lambda = \oint_{C_\lambda} \underline{p} \cdot d\underline{q}$ ≡ action computed at fixed value of λ is adiabatic invariant

→ adiabatic invariant is C.O.M. on time scales $\tau > \omega^{-1}$.

→ adiabatic invariant is intrinsically / implicitly referenced to a given time scale. Same system can manifest multiple adiabatic invariants on different time scales.

→ $I = \oint_C \underline{p} \cdot d\underline{q}$ → Poincare - Cartan Invariant

→ exact COM

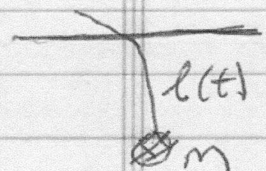
to calculate explicitly, need integrable motion (as in explicit representation of action-angle var.)

but

— $I_\lambda = \oint_{C_\lambda} \underline{p}\cdot d\underline{q}$ is <u>approximation</u> to I,

computed for <u>fixed λ</u>. $\dot{I}_\lambda \cong 0$ for $t >> \omega^{-1}$.

<u>Examples</u>: i) <u>Pendulum — the prototype</u>



$$\frac{\dot{l}(t)}{l} < \sqrt{g/l}$$

How does Θ vary with l?

$I = E/\omega$, understood $E = \overline{E}$

$\omega = \sqrt{g/l}$

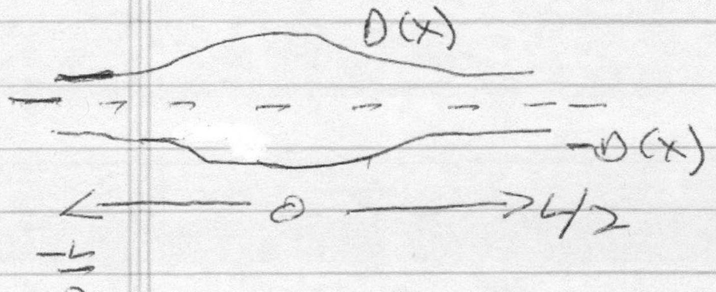
$$\overline{E} = \frac{m}{2} l^2 \overline{\dot{\Theta}^2} + mgl\frac{\overline{\Theta^2}}{2}$$

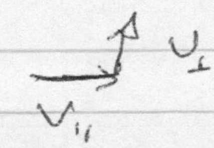
$$= mgl\overline{\Theta^2}$$

$$I = m\sqrt{g}\, l^{3/2}\, \overline{\Theta^2}$$

so $\Theta_{rms} \sim l^{-3/4}$ i.e. amplitude decreases as length increases more generally, $\dfrac{\theta_{rms}(t)}{\theta_{rms}(0)} \sim \left(\dfrac{\ell(0)}{\ell(t)}\right)^{3/4}$

2.) Mechanical Mirror





$\tau_{b_\perp} \sim \left(v_\perp / 2D\right)^{-1} \rightarrow \perp$ bounce time

$\tau_b \ll L / v_\parallel$

many bounces($\perp$) in time to sense curvature of D.

now,

$$2\pi I = \int_{-D}^{D} m v_\perp \, dy + \int_{D}^{-D} (-m v_\perp) \, dy$$

$$= 4 m D v_\perp$$

$$\boxed{I = \frac{2}{\pi} D m v_\perp}$$ Adiabatic Invariant

$$E = \frac{1}{2} m \left(v_\perp^2 + v_\parallel^2\right)$$

More on Adiabatic Invariants

→ for parameter $\lambda(t)$ s/t

$$\left.\dot{\lambda}(t)/\lambda < \omega\right] \rightarrow \text{multiple time scale.}$$

$$\frac{d}{dt}\bar{I} = 0 \qquad \bar{I} = \oint p\,dq$$

$E\ \lambda$ fixed

$\bar{I}$ → adiabatic invariant

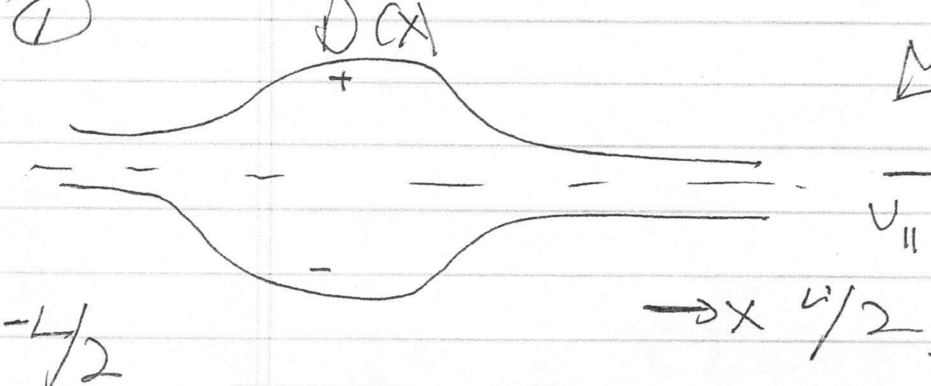
→ adiabatic invariance ⟺ phase symmetry, along $\oint$.

(i.e. can start anywhere in integration).

Applications of Adiabatic Invariants

Consider 2 related non-trivial (adiabatic invariant-related) systems:

① Mechanical Mirror

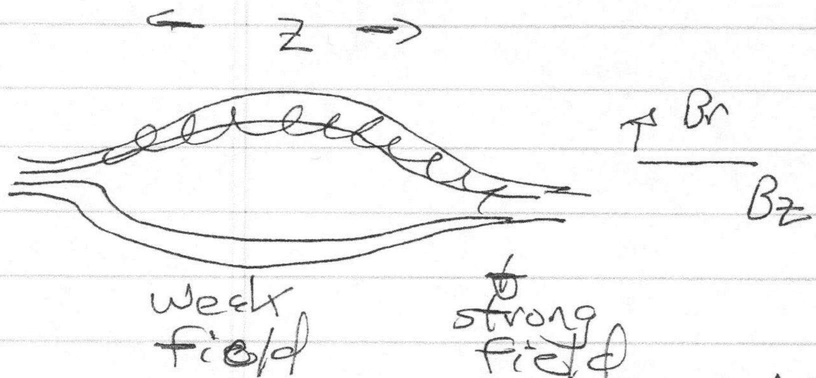


(i.e. curved hard surface)

cf: video

n.b. $\boxed{D/L \ll 1.}$

② Magnetic Mirror → basis for mechanical mirror.



$\nabla \cdot B = 0$

$$\frac{\partial}{\partial z} B_z + \nabla_r B_r = 0$$

i.e. $\frac{\partial}{\partial z} B_z \neq 0 \Rightarrow \partial_r B_r \neq 0$

for "long, thin" mirror — anisotropy! ⇒ { long thin ⇒ slow axial variation

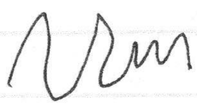
$$B_r \simeq -\frac{r}{2}\frac{\partial B_z}{\partial z}\bigg|_{r_0}$$

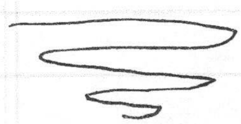
from: $B_r = -\frac{1}{r}\int_0^r dr'\, r' \frac{\partial B_z}{\partial z}$

Consider time scales:

$\rightarrow \tau_{b_\perp} \sim (V_\perp/2D)^{-1} \Rightarrow \perp$ bounce time

$\rightarrow \tau_{b_\parallel} \sim L/V_\parallel \Rightarrow$ parallel bounce time

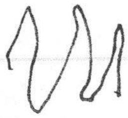
i.e. $\perp$ 

$\parallel$ 

so if consider

$\tau_{b_\perp} < t \Rightarrow$
- many bounces.
- sufficient time to sense curvature of D
- can define adiabatic invariant

$$\int p_\perp dq_\perp$$

$$2\pi I = \oint m V_\perp dy \rightarrow \oint p_\perp dq_\perp$$ 

$$= \int_{-D}^{D} dy \underbrace{mV_\perp}_{\text{forward}} + \int_{D}^{-D} \underbrace{(-mV_\perp)}_{\text{back.}} dy$$

$$= 4mDV_\perp$$

$$\boxed{I = \frac{2}{\pi} D m V_\perp}$$

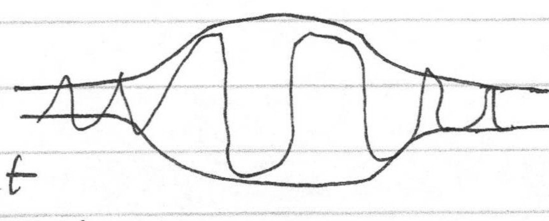
adiabatic invariant on times $t > \tau_{b_\perp}$

i.e. $D\,V_\perp \sim$ const

$V_\perp \begin{cases} \text{large in throat} \\ \text{smaller in center} \end{cases}$

given initial $D(x_0)\,V_\perp(x_0)$, can determine $V_\perp(x)$ for all x.

Motion ?



$\left\{ \begin{array}{l} \text{particle can} \\ \text{reflect from throat} \end{array} \right.$

energy conserved !

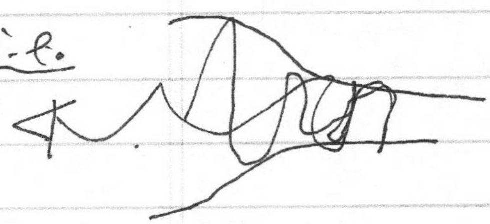
$$E = \tfrac{1}{2} m \left(V_\perp^2 + V_\parallel^2\right)$$

$$= \tfrac{1}{2} m \left(V_\parallel^2 + \frac{\pi^2}{4D^2} \frac{I^2}{m^2} \right)$$

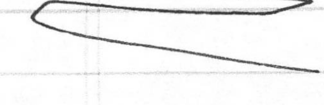
$$\Rightarrow \quad V_\parallel^2 = \frac{2E}{m} - \frac{\pi^2}{4D(x)^2} \frac{I^2}{m^2}$$

so if I s/t $\dfrac{\pi^2 I^2}{4D(x)^2 m^2} > \dfrac{2E}{M} \Rightarrow$ particle reflected in mirror throat.

i.e.



$$I = \frac{2}{\pi} D(x_0)\, m V_{\perp 0}$$



frequently written as:

$$I = \frac{2}{\pi} D(0)\, m V_\perp(0)$$

$x_0 \Leftrightarrow$ center.

$$\frac{\pi^2 I^2}{4 D(x)^2 M^2} > \frac{2E}{m}$$

i.e.

$$\Rightarrow \left(\frac{D(x_0)}{D(x)}\right)^2 v_\perp^2(x_0) > \frac{2E}{m}.$$

i.e.

for $x < L$ ⇒ particle will bounce.

As $E = \frac{1}{2} m \left(v_{\perp 0}^2 + v_{\parallel 0}^2\right)$,

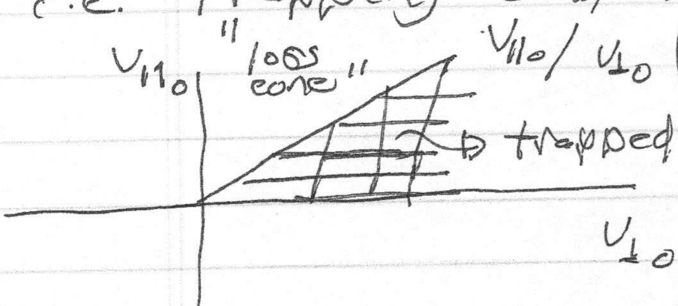
$$\Rightarrow \boxed{\frac{v_{\parallel 0}^2}{v_{\perp 0}^2} < \left(\frac{D(0)}{D(L)}\right)^2 - 1}$$

"mirror ratio"

i.e. optimal ratio

$$R_M \equiv \frac{D(x_0)^2}{D(x)^2} \to \frac{D(0)^2}{D(L)^2}$$

i.e. trapping condition



(for $x_0 \approx 0$, $x \approx L$)

$$R_M \equiv \frac{D(0)^2}{D(L)^2}$$

basic description of mirror confinement

Now, can determine reflection point

simply by:

$$v_{\parallel}^2 = \frac{2E}{m} - \frac{\pi^2}{4D(x_R)^2}\frac{I^2}{m^2} = 0$$

$\downarrow$ defines

$x_R \leq L/2$

then: can envision longer times:

$$t > \tau_{b_{\parallel}} \gg \tau_{b_{\perp}}$$

$$\tau_{b_{\parallel}} = \oint \frac{dx}{|v_{\parallel}|}$$

$\downarrow$

parallel bounce time, for trapped particles

so

Can have "2nd" adiabatic invariant on time scale $\tau_{b_{\parallel}} > \tau_{b_{\perp}}$

$$J_{\parallel} = \oint dx\, p_{\parallel}$$

$\downarrow$

"bounce invariant"

2nd.

$J_{\perp} \Rightarrow$ first adiabatic inv.

$\Rightarrow \perp$ bounce.

i.e.

$$\tau_\perp < \tau < \tau_\parallel < \tau'$$

$\perp$ bounce time

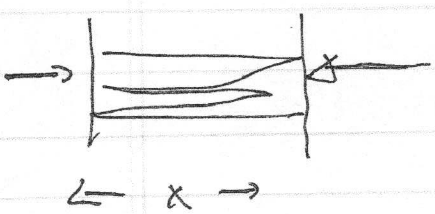
time of $I = J$ invariant

parallel bounce time

time of $J_\parallel$ invariant

N.B.: Can expect 1 adiabatic invariant per closed cyclic orbit (n.b. cyclic orbit in action-angle sense).

For application of $J_\parallel$: [Adiabatic compression]



if push slowly:

$$J_\parallel = \oint p_\parallel \, dx = \text{const}$$

$$J_\parallel = \int_{-L}^{L} p_\parallel \, dx + \int_{L}^{-L} -p_\parallel \, dx$$

$$= p_\parallel (2L) - - p_\parallel (2L)$$

$$\delta J_\parallel = 0 \Rightarrow \delta (p_\parallel L) = 0$$

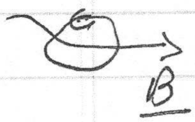
$$\Rightarrow \delta p_\parallel = - \delta L$$

(2) Magnetic Mirror

→ scheme is the same, with magnetic field variation as agent of confinement

→ now, for particle in magnetic field

$$\underline{p} \rightarrow \underline{p} - \frac{e}{c}\vec{A}$$



consider cyclotron orbit in plane ⊥ to field

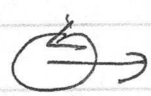
$$\oint_{\perp \text{ plane}} p \, d\underline{l} = \oint_{\text{cycl.}} \underline{p}_\perp \cdot d\underline{\ell}_\perp$$

⇒ integrated along Larmor orbit.

$$= \int_c \underline{p}_\perp \cdot d\underline{\ell}_\perp - \frac{e}{c}\int_c A_\perp \cdot d\underline{\ell}_\perp$$

$$= \int_c m \underline{v}_\perp \cdot d\underline{\ell}_\perp - \frac{e}{c}\int_c A_\perp \cdot d\underline{\ell}_\perp$$

Larmor disk



$$= m v_\perp (\rho_L 2\pi) - \frac{e}{c}\pi \rho_L^2 B$$

↓ $2\pi r$ with r = radius of Larmor disk

↳ flux thru Larmor disk.

so

$$\oint_{\perp} p\,dq = m v_\perp \frac{v_\perp}{\frac{eB}{mc}} 2\pi - \frac{e}{c} \pi B \frac{v_\perp^2}{\frac{e^2 B^2}{m^2 c^2}}$$

$$= \frac{m v_\perp^2}{2B}\left(4\pi \frac{Mc}{|e|}\right) - \frac{m v_\perp^2}{2B}\left(2\pi \frac{Mc}{|e|}\right)$$

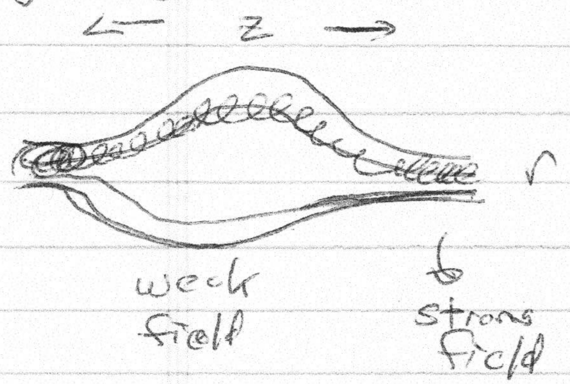
$$= \frac{m v_\perp^2}{2B}\underbrace{\left(\frac{4\pi Mc}{|e|}\right)}_{\text{irrelevant const.}}$$

$$\underline{\underline{\text{so}}} \quad \oint_{\perp} p\,dq = \underbrace{\frac{m v_\perp^2}{2B}}_{\text{magnetic moment}_{\mu}}.$$

Physically : — Magnetic moment corresponds to action computed for 1 cyclotron orbit

— adiabatic invariant on $t > \tau_{cycl}$, else approx. of closure of cyclotron orbit is meaningless.

3.) Magnetic Mirror – basis for mechanical mirror



$\underline{\nabla} \cdot \underline{B} = 0$

$\frac{\partial}{\partial z} B_z + \nabla_r B_r = 0$

$\neq 0$

Now, consider rate of change of $\perp$ Energy

$$\frac{d}{dt}\left(\frac{m v_\perp^2}{2}\right) = q\, \underline{E}_\perp \cdot \underline{v}_\perp$$

avg over 1 cyclotron orbit $\Rightarrow$

$$\delta\left(\frac{m v_\perp^2}{2}\right) = \int_{\Omega^{-1}} dt\, q\, \underline{E}_\perp \cdot \underline{v}_\perp$$

$\underline{v}\, dt = \underline{\ell}$

change in energy in 1 cyclotron orbit

$$= \int_{\substack{\rho \\ \text{gyro circle}}} d\underline{\ell} \cdot \underline{E}_\perp q = q \int \underline{E} \cdot d\underline{\ell}$$

$\rho \to$ gyro-radius

$$= \int d\underline{a}\, q \cdot \underline{\nabla} \times \underline{E}$$

via Faraday.

$$= -\int_{\text{gyrodisk}} d\underline{a} \cdot \left(\frac{q}{c} \frac{\partial B}{\partial t}\right)$$

$$\cong -\pi \rho^2 \frac{q}{c} \frac{\partial B}{\partial t}$$

10.

$\rho^2 = V_\perp^2/\Omega^2$

$\Rightarrow$

$$\delta\left(\frac{mV_\perp^2}{2}\right) \cong -\pi \frac{e}{c} \frac{V_\perp^2}{\frac{e^2 B^2}{m^2c^2}} \frac{\partial B}{\partial t}$$

$$= -\frac{mV_\perp^2}{\Omega} \frac{\pi}{B} \frac{\partial B}{\partial t}$$

but $\underbrace{\delta B}_{\substack{\text{change in} \\ \text{1 cyclotron} \\ \text{period}}} = \underbrace{\frac{2\pi}{\Omega}}_{\tau_c} \frac{\partial B}{\partial t}$

$\therefore$

$$\delta\left(\frac{mV_\perp^2}{2}\right) = -\frac{mV_\perp^2}{2} \frac{1}{B} \delta B$$

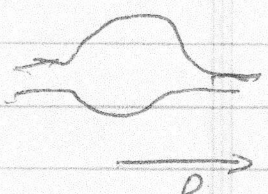
$\Rightarrow$ $\boxed{\delta\left(\frac{mV_\perp^2}{2B}\right) = 0}$ $\Rightarrow$ adiabatic time variation in $B \Rightarrow$ heating

so $\boxed{\mu = mV_\perp^2/2B}$ $\rightarrow$ magnetic moment adiabatic invariant on $t \gg \Omega^{-1}$

Now, for mirroring:

$$\cancel{\tfrac{1}{2}m}\left(V_\parallel^2 + V_\perp^2\right) = \cancel{\tfrac{1}{2}m}\left(V_{\parallel 0}^2 + V_{\perp 0}^2\right)$$

$$\frac{\cancel{m}\, V_\perp^2(0)}{\cancel{2}B(0)} = \frac{\cancel{m}\, V_\perp^2(\ell)}{\cancel{2}B(\ell)}$$



$$V_\parallel^2(0) + V_\perp^2(0) = V_\parallel^2 + \frac{B(\ell)}{B(0)} V_\perp^2(0)$$

$$V_\perp^2(0)\left(1 - \frac{B(\ell)}{B(0)}\right) = V_\parallel^2(\ell) - V_\parallel^2(0)$$

for confinement: $V_\parallel^2(\ell) = 0 \Rightarrow$

so

$$\boxed{\frac{V_\parallel^2(0)}{V_\perp^2(0)} < \underbrace{\frac{B(\ell)}{B(0)}}_{\text{mirror ratio}} - 1}$$

obvious analogy to:

$$\frac{V_{\parallel 0}^2}{V_{\perp 0}^2} < \frac{D(x_0)^2}{D(x_R)^2} - 1$$

$D(\ell) \leftrightarrow 1/D(x_R)$ → strong B → frequent gyration, frequent bouncing

$B(0) \leftrightarrow 1/D(x_0)$ → weak B → less frequent bouncing, gyration.

12.

Similarly, can define bounce invariant:

$$J_{\parallel} = \oint d\ell \left[2m\left(E - \mu B(\ell)\right)\right]^{1/2}$$

longitudinal action

i.e. $V_{\parallel}^2(\ell) = V_{\parallel}^2(0) + V_{\perp}^2(0) - \mu B(\ell)$

etc.

squeeze → energy gain

N.B.: Treatment of adiabatic invariants given here corresponds to lowest order p.t. in $\frac{1}{\lambda}\frac{d\lambda}{dt}\Big/\omega < 1$

$$\underbrace{\frac{1}{\lambda}\frac{d\lambda}{dt}\Big/\omega}_{\text{"}O(\epsilon)\text{"}}$$ here.

Note: Can also define 'mirror force',

$v_r \quad v_\theta \quad v_z$

$B_r \quad B_\theta \quad B_z$

$$F = \frac{q}{c}\,\underline{v}\times B$$

$$F_z = \frac{q}{c}\left(\cancel{v_r B_\theta} - v_\theta B_r\right)$$

$$\cong \frac{q}{c}\,\frac{v_\theta}{2}\,\frac{r\,\partial B_z}{\partial z}$$

$v_\theta \to v_\perp$

$r \to \rho_L$

13.

$$F_z \cong \frac{I}{c} \frac{v_\theta r}{2} \frac{\partial B_z}{\partial z}$$

$$= \pm \frac{m v_\perp^2}{2B} \frac{\partial B}{\partial z} = \mp \mu \frac{\partial B}{\partial z}$$

$\left\{ \begin{array}{l} \text{depends on location} \\ \text{in trajectory} \end{array} \right.$